q-Peano Kernel and Its Applications

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Abstract

We introduce a q-analogue of the Peano kernel theorem by replacing ordinary derivatives and integrals by quantum derivatives and quantum integrals. In the limit $q \to 1$, the q-Peano kernel reduces to the classical Peano kernel. We also give applications to polynomial interpolation and construct examples in which classical remainder theory fails whereas q-Peano kernel works. Furthermore we derive a relation between q-B-splines and divided differences via the q-Peano kernel.
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1 Introduction

The Peano kernel theorem provides a useful technique for computing the errors of approximations such as interpolation, quadrature rules and B-splines. The errors are represented by a linear functional that operates on functions $f \in C^{n+1}[a,b]$ and annihilates all polynomials of degree at most n.

Namely, if $L(f) = 0$ for all $f \in \mathscr{P}_n$, the space of polynomials of degree n, then

$$L(f) = \int_a^b f^{(n+1)}(t)K(x,t)dt,$$

where $K(x,t) = \dfrac{1}{n!}L\left((x-t)_+^n\right)$.

An important application of this result is the Kowalewski's interpolating polynomial remainder. Let $t_0, t_1, \ldots, t_n \in [a,b]$ be fixed and distinct, and

$$L(f) = f(x) - \sum_{k=0}^{n} f(t_k) l_{nk}(x)$$

where $l_{nk}(x) = \prod_{\substack{v=0 \\ v\neq k}}^{n} \dfrac{x-t_v}{t_k-t_v}$. If $f \in C^{m+1}[a,b]$, then

$$L(f) = \frac{1}{m!}\sum_{k=0}^{n} l_{nk}(x)\int_{t_k}^{x}(t_k-t)^m f^{(m+1)}(t)dt, \quad \text{for each } m = 0,1,\ldots,n$$

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is the error functional, see [5]. Our purpose is to extend the Peano kernel when classical derivatives are replaced by q-derivatives. This extension is important because there are functions whose q-derivatives exist but whose classical derivatives fail to exist.

Section 2 contains definitions and properties of the quantum calculus which we use in the next sections. In Section 3, we give the q-Taylor theorem and develop a q-analogue of the Peano kernel (q-Peano kernel). Furthermore, we present a simple way to find the kernel under some conditions. Section 4 demonstrates how the q-Peano kernel is used to find the error of Lagrange interpolation. A q-analogue of the trapezoidal rule is also given. Moreover, we discuss the error bounds of quadrature formula on the remainder. Finally, we establish a relation between the q-B-splines and the q-Peano kernel in Section 5.

2 Preliminaries

We begin by giving basic definitions and theorems of the q-calculus that are required in the next section. For a fixed parameter $q \neq 1$, the q-derivatives are defined by,

$$\begin{aligned} D_q f(t) &= \frac{f(qt)-f(t)}{(q-1)t} \\ D_q^n f(t) &= D_q(D_q^{n-1} f(t)), \quad n \geqslant 2. \end{aligned}$$

Note that if f is a differentiable function, then

$$\lim_{q\to 1} D_q f(x) = Df(x).$$

For polynomials the q-derivative is easy to compute. Indeed it follows easily from the definition of the q-derivative that

$$D_q x^n = [n]_q x^{n-1},$$

where the q-integers $[n]_q$ are defined by,

$$[n]_q = \begin{cases} (1-q^n)/(1-q), & q \neq 1, \\ n, & q = 1. \end{cases}$$

Moreover, the q-factorial is defined by

$$[n]_q! = [1]_q \cdots [n]_q.$$

Quantum integrals are the analogues of classical integrals for the quantum calculus. Quantum integrals satisfy a quantum version of the fundamental theorem of calculus, see [7] for details.

Definition 2.1. *Let $0 < a < b$. Then the definite q-integral of a function $f(x)$ is defined by*

$$\int_0^b f(x) d_q x = (1-q) b \sum_{i=0}^{\infty} q^i f(q^i b)$$

and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

Theorem 2.2. *[Fundamental Theorem of Calculus]*
If $F(x)$ is continuous at $x = 0$, then

$$\int_a^b D_q F(x) d_q x = F(b) - F(a)$$

where $0 \leqslant a < b \leqslant \infty$.

The work [11] gives the mean value theorem in the q-calculus which will be needed in one of our results.

Theorem 2.3. *If F is continuous and G is $1/q$-integrable and is nonnegative(or nonpositive) on $[a,b]$, then there exists $\tilde{q} \in (1,\infty)$ such that for all $q > \tilde{q}$ there exists a $\xi \in (a,b)$ for which*

$$\int_a^b F(x)G(x)d_{1/q}x = F(\xi)\int_a^b G(x)d_{1/q}x.$$

We also require a q-Hölder inequality and appropriate notions of distance in q-integrals, see [2], [4] and [13].

Definition 2.4. *We will denote by $L_{p,q}([0,b])$ with $1 \leqslant p < \infty$ the set of all functions f on $[0,b]$ such that*

$$||f||_{p,q} := \left(\int_0^b |f|^p d_{1/q}t\right)^{\frac{1}{p}} < \infty.$$

Furthermore let $L_{\infty,q}([0,b])$ denote the set of all functions f on $[0,b]$ such that

$$||f||_{\infty,q} := \sup_{x\in[0,b]} |f(x)| < \infty.$$

Theorem 2.5. *Let $x \in [0,b]$, $q \in [1,\infty)$ and $p_1, p_2 > 1$ be such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Then*

$$\int_0^x |f(x)||g(x)|d_{1/q}t \leqslant \left(\int_0^x |f(x)|^{p_1} d_{1/q}t\right)^{\frac{1}{p_1}} \left(\int_0^x |g(x)|^{p_2} d_{1/q}t\right)^{\frac{1}{p_2}}.$$

3 q-Peano Kernel Theorem

In this section we derive a generalization of the Peano kernel theorem. This generalization is based on the q-Taylor expansion analogous to the proof of the classical Peano kernel Theorem. So we start by giving the q-Taylor expansion with integral representation. A detailed treatment of the classical Peano Kernel theorem can be found in [5], [9] and [10].

We use the notation q-$C^k[a,b]$ to denote the space of functions whose q-derivatives of order up to k are continuous on $[a,b]$.

Theorem 3.1. *(q-Taylor Theorem) Let f be $n+1$ times $1/q$-differentiable in the closed interval $[a,b]$. Then*

$$f(x) = \sum_{k=0}^{n} q^{k(k-1)/2} \frac{(D_{1/q}^k f)(q^k a)}{[k]_q!}(x-a)^{k,q} + R_n(f), \tag{1}$$

where

$$(x-t)^{n,q} = (x-q^{n-1}t)\cdots(x-qt)(x-t)$$

and

$$R_n(f) = \frac{q^{n(n+1)/2}}{[n]_q!}\int_a^x (D_{1/q}^{n+1} f)(q^n t)(x-t)^{n,q} d_{1/q}t.$$

Another way to express the remainder $R_n f$ is to employ the truncated power function. That is

$$R_n(f) = \frac{q^{n(n+1)/2}}{[n]_q!}\int_a^b (D_{1/q}^{n+1} f)(q^n t)(x-t)_+^{n,q} d_{1/q}t, \tag{2}$$

where

$$(x-t)_+^{n,q} = (x-q^{n-1}t)\cdots(x-qt)(x-t)_+.$$

Here $(x-t)_+$ is the truncated power function

$$(x-t)_+ = \begin{cases} x-t, & \text{if} \quad x>t \\ 0, & \text{otherwise.} \end{cases}$$

There are other forms of q-Taylor Theorem, see for example [1], [8], [6].

Theorem 3.2. *Let $g_t(x) = (x-t)_+^{n,q}$ and let L be a linear functional that commutes with the operation of q-integration and also satisfies the conditions: $L(g_t)$ exists and $L(f) = 0$ for all $f \in \mathscr{P}_n$. Then for all $f \in 1/q - C^{n+1}[a,b]$*

$$L(f) = \int_a^b (D_{1/q}^{n+1} f)(q^n t) K(x,t) d_{1/q} t,$$

where

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!} L(g_t).$$

Proof. Recall that here the function $(x-t)_+^{n,q}$ is a function of t and x behaves as a parameter. When we say $L(g_t)$ we mean that L is applied to the truncated power function, regarded as a function of x with t as a parameter. Hence we find real number that depends on t. We apply L to the equation (1). Since L is linear and annihilates polynomials, we have

$$L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} L\left(\int_a^b (D_{1/q}^{n+1} f)(q^n t)(x-t)_+^{n,q} d_{1/q} t\right).$$

Since L commutes with the operation of q-integration,

$$L(f) = \frac{q^{n(n+1)/2}}{[n]_q!} \int_a^b (D_{1/q}^{n+1} f)(q^n t) L\left((x-t)_+^{n,q}\right) d_{1/q} t.$$

□

Corollary 3.3. *If the conditions in Theorem 3.2 are satisfied and also the kernel $K(x,t)$ does not change sign on $[a,b]$, then*

$$L(f) = \frac{\left(D_{1/q}^{n+1} f\right)(\xi)}{[n+1]_q!} q^{n(n+1)/2} L(x^{n+1})$$

Proof. Since $D_{1/q}^{n+1} f$ is continuous and $K(x,t)$ does not change sign on $[a,b]$, we can apply the Mean Value Theorem 2.3. Thus we have

$$L(f) = \left(D_{1/q}^{n+1} f\right)(\xi) \int_a^b K(x,t) d_{1/q} t, \quad a < \xi < b.$$

Replacing $f(x)$ by x^{n+1} gives

$$L(x^{n+1}) = \frac{[n+1]_q!}{q^{n(n+1)/2}} \int_a^b K(x,t) d_{1/q} t,$$

so

$$\int_a^b K(x,t) d_{1/q} t = \frac{q^{n(n+1)/2}}{[n+1]_q!} L(x^{n+1}),$$

and this completes the proof. □

4 Application to polynomial interpolation

The main idea in this section is to apply the q-Peano kernel Theorem on the remainder of polynomial interpolation. Findings demonstrate the advantage of using the q-Peano kernel Theorem where the classical theorem does not work.

Proposition 4.1. *Suppose $t_0, t_1, \ldots, t_n \in [a,b]$ are distinct points. For a fixed $x \in [a,b]$, define the corresponding error functional by*

$$L(f) = f(x) - \sum_{k=0}^{n} f(t_k) l_{nk}(x).$$

Then

$$L(f) = \frac{q^{m(m+1)/2}}{[m]_q!} \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t \quad \text{for each } m = 0, 1, \ldots, n.$$

Proof. Since $\sum_{k=0}^{n} l_{nk}(x) = 1$, by the q-Peano kernel Theorem 3.2 we get,

$$\begin{aligned} \frac{[m]_q!}{q^{m(m+1)/2}} K(x,t) = L\left((x-t)_+^{m,q}\right) &= (x-t)_+^{m,q} - \sum_{k=0}^{n} (t_k - t)_+^{m,q} l_{nk}(x) \\ &= \sum_{k=0}^{n} \left[(x-t)_+^{m,q} - (t_k - t)_+^{m,q}\right] l_{nk}(x). \end{aligned}$$

From the fact that

$$\begin{aligned} \int_a^b \left[(x-t)_+^{m,q} - (t_k - t)_+^{m,q}\right] \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t = &\int_a^x \left[(x-t)^{m,q} - (t_k - t)^{m,q}\right] \left(D_{1/q}^{m+1} f\right)(q^m t) \\ &+ \int_{t_k}^{x} (t_k - t)^{m,q} \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t \end{aligned}$$

we have

$$\begin{aligned} \frac{[m]_q!}{q^{m(m+1)/2}} \int_a^b K(x,t) \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t = &\int_a^x \left(D_{1/q}^{m+1} f\right)(q^m t) \sum_{k=0}^{n} \left[(x-t)^{m,q} - (t_k - t)^{m,q}\right] l_{nk}(x) d_{1/q}t \\ &+ \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t. \end{aligned}$$

For each $m \leqslant n$, since the interpolation operator is a projection, it reproduces polynomials and the term in the square brackets vanishes in the last equation for $f(x) = (x-t)^{m,q}$. Accordingly,

$$\begin{aligned} L(f) &= \int_a^b K(x,t) \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t \\ &= \frac{q^{m(m+1)/2}}{[m]_q!} \sum_{k=0}^{n} l_{nk}(x) \int_{t_k}^{x} (t_k - t)^{m,q} \left(D_{1/q}^{m+1} f\right)(q^m t) d_{1/q}t \quad \text{for each } m = 0, 1, \ldots, n. \end{aligned}$$

□

Now we give examples that show how we can find the q-Peano kernel.

Example 4.2. *Suppose that we interpolate a function $f \in 1/q - C^3[-1,1]$ by a polynomial $p \in \mathscr{P}_2$. Here $n = 2$ and $m = 2$. Let $t_0 = -1,\ t_1 = 0,\ t_2 = 1$. Then the error function becomes*

$$L(f) = \frac{q^3}{[2]_q!} \sum_{k=0}^{2} l_{2k}(x) \int_{t_k}^{x} (t_k - t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t$$

with $l_{20}(x) = \frac{1}{2}x(x-1)$, $l_{21}(x) = (1-x^2)$, $l_{22}(x) = \frac{1}{2}x(x+1)$. Then,

$$\begin{aligned}\frac{[2]_q!}{q^3} L(f) = \quad & l_{20}(x) \int_{-1}^{x} (-1-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t + l_{21}(x) \int_{0}^{x} (-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t \\ & + l_{22}(x) \int_{1}^{x} (1-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t.\end{aligned}$$

Now if $x \leqslant 0$, then

$$\begin{aligned}\frac{[2]_q!}{q^3} L(f) = \quad & l_{20}(x) \int_{-1}^{x} (-1-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t - l_{21}(x) \int_{x}^{0} (-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t \\ & - l_{22}(x) \int_{x}^{0} (1-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t - l_{22}(x) \int_{0}^{1} (1-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t.\end{aligned}$$

Hence,

$$L(f) = \frac{q^3}{[2]_q!} \int_{-1}^{1} K(x,t) \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t$$

where

$$K(x,t) = \begin{cases} l_{20}(x)(-1-t)^{2,q}, & -1 \leqslant t \leqslant x \\ -l_{21}(x)(-t)^{2,q} - l_{22}(x)(1-t)^{2,q}, & x \leqslant t \leqslant 0 \\ -l_{22}(x)(1-t)^{2,q}, & 0 \leqslant t \leqslant 1. \end{cases}$$

Similarly for $x \geqslant 0$,

$$\begin{aligned}\frac{[2]_q!}{q^3} L(f) = \quad & l_{20}(x) \int_{-1}^{0} (-1-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t + l_{20}(x) \int_{0}^{x} (-1-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t \\ & + l_{21}(x) \int_{0}^{x} (-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t - l_{22}(x) \int_{x}^{1} (1-t)^{2,q} \left(D^3_{1/q} f\right)(q^2 t) d_{1/q} t\end{aligned}$$

and the Peano kernel becomes

$$K(x,t) = \begin{cases} l_{20}(x)(-1-t)^{2,q}, & -1 \leqslant t \leqslant 0 \\ l_{20}(x)(-1-t)^{2,q} + l_{21}(x)(-t)^{2,q} - l_{22}(x)(1-t)^{2,q}, & 0 \leqslant t \leqslant x \\ -l_{22}(x)(1-t)^{2,q}, & x \leqslant t \leqslant 1. \end{cases}$$

Example 4.3. *Let*

$$f(x)=\begin{cases}\dfrac{q^3x^3}{6}, & 0\leqslant x<1\\[2mm] \dfrac{1}{6}\left(4-4[3]_qx+4q[3]_qx^2-3q^3x^3\right), & 1\leqslant x<2\\[2mm] \dfrac{1}{6}\left(-44+20[3]_qx-8q[3]_qx^2+3q^3x^3\right), & 2\leqslant x<3\\[2mm] -\dfrac{1}{6}(-4+x)(-4+qx)(-4+q^2x), & 3\leqslant x<4\\[2mm] 0, & \textit{otherwise.}\end{cases}$$

It is obvious that for $q\neq 1$, $f\in C[0,4]$ but $f\notin C^1[0,4]$. However, one may check that $f\in 1/q-C^2[0,4]$. Classical error functionals cannot work but we may find the error via the q-Peano Kernel theorem. Let $t_0=0$, $t_1=2$ and $t_2=4$. Then the error functional

$$L(f)=q\sum_{k=0}^{2}l_{2k}(x)\int_{t_k}^{x}(t_k-t)\left(D_{1/q}^2f\right)(qt)d_{1/q}t$$

where $l_{20}(x)=\frac{1}{8}(x-2)(x-4)$, $l_{21}(x)=-\frac{1}{4}x(x-4)$ and $l_{22}(x)=\frac{1}{8}x(x-2)$. Then,

$$\begin{aligned}\frac{1}{q}L(f)= \quad & l_{20}(x)\int_0^x(-t)\left(D_{1/q}^2f\right)(qt)d_{1/q}t+l_{21}(x)\int_2^x(2-t)\left(D_{1/q}^2f\right)(qt)d_{1/q}t\\ &+l_{22}(x)\int_4^x(4-t)\left(D_{1/q}^2f\right)(qt)d_{1/q}t.\end{aligned}$$

Now we will find the kernel. If $0\leqslant x<2$, then

$$K(x,t)=\begin{cases}-l_{20}(x)t, & 0\leqslant t<x\\ l_{21}(x)(2-t)-l_{22}(x)(4-t), & x\leqslant t<2\\ -l_{22}(x)(4-t), & 2\leqslant t<4.\end{cases}$$

Similarly, for $2\leqslant x<4$,

$$K(x,t)=\begin{cases}-l_{20}(x)t, & 0\leqslant t<2\\ -l_{20}(x)t+l_{21}(x)(2-t), & 2\leqslant t<x\\ l_{21}(x)(2-t)-l_{22}(x)(4-t), & x\leqslant t<4.\end{cases}$$

The function $f(x)$ given above is indeed a cubic q-B-spline. q-B-splines form a basis for quantum splines which are piecewise polynomials whose quantum derivatives agree up to some order at the joins, see [12].

4.1 Trapezoidal rule in q-integration

Consider the $1/q$-integral of a function f on the interval $[a,b]$. We want to evaluate the q-integral approximately using linear interpolant formula. That is,

$$\int_a^b f(x)d_{1/q}x \approx \frac{b-aq}{[2]_q}f(a) + \frac{bq-a}{[2]_q}f(b)$$

Let us define the operator L as

$$L(f) = \int_a^b f(x)d_{1/q}x - \frac{b-aq}{[2]_q}f(a) - \frac{bq-a}{[2]_q}f(b).$$

Since $L(f) = 0$ for all functions $f \in \mathscr{P}_1$, for all $f \in 1/q - C^2[a,b]$ we have

$$L(f) = \int_a^b \left(D^2_{1/q}f\right)(qt)K(x,t)d_{1/q}t$$

and

$$K(x,t) = qL((x-t)_+).$$

What follows we find the kernel $K(x,t)$. First,

$$K(x,t) = q\left\{\int_a^b (x-t)_+ d_{1/q}x - \frac{b-aq}{[2]_q}(a-t)_+ - \frac{bq-a}{[2]_q}(b-t)_+\right\}.$$

Then for $t \in [a,b]$,

$$\int_a^b (x-t)_+ d_{1/q}x = \int_t^b (x-t)d_{1/q}x, \quad (a-t)_+ = 0 \quad \text{and} \quad (b-t)_+ = (b-t)$$

Thus,

$$\begin{aligned} K(x,t) &= q\left\{\int_t^b (x-t)d_{1/q}x - \frac{bq-a}{[2]_q}(b-t)\right\} \\ &= q\left\{\frac{(b-t)(b-\frac{t}{q})}{[2]_{1/q}} - \frac{bq-a}{[2]_q}(b-t)\right\} \\ &= \frac{q}{[2]_q}(b-t)(a-t) \end{aligned}$$

for $a \leqslant t \leqslant b$.
Notice that $K(x,t) < 0$ on $[a,b]$. Then we can apply Mean Value Theorem 2.3. So, we have

$$L(f) = \frac{D^2_{1/q}f(\xi)}{[2]_q!}qL(x^2),$$

where

$$\begin{aligned} L(x^2) &= \int_a^b x^2 d_{1/q}x - \frac{b-aq}{[2]_q}a^2 - \frac{bq-a}{[2]_q}b^2 \\ &= \frac{b^3}{[3]_q!} - \frac{a^3}{[3]_q!} - \frac{b-aq}{[2]_q}a^2 - \frac{bq-a}{[2]_q}b^2 \\ &= \frac{-(b-a)(bq-a)(b-aq)}{[3]_q!} \end{aligned}$$

Finally, we derive

$$\begin{aligned} L(f) &= \int_a^b f(x)d_{1/q}x - \frac{b-aq}{[2]_q}f(a) - \frac{bq-a}{[2]_q}f(b) \\ &= \frac{-q(b-a)(bq-a)(b-aq)}{[3]_q![2]_q!}D^2_{1/q}f(\xi) \end{aligned}$$

where $a < \xi < b$.
When $q = 1$, the above equation reduces to the well-known trapezoidal rule, see [9].

4.2 Remainder on quadrature

We now discuss error bounds of quadrature formulas on remainders given by

$$R_n(f;q) = \int_0^b f(x)d_{1/q}x - \sum_{k=0}^{n} \gamma_{nk} f(t_{nk})$$

which appear in numerical integration. Assuming $f \in 1/q - C^{m+1}[0,b]$ and $R_n(f;q) = 0$ for all $f \in \mathscr{P}_m$, we can apply the q-Peano kernel theorem. Hence

$$R_n(f;q) = \int_0^b K(x,t)\left(D^{m+1}_{1/q}f\right)(q^m t)d_{1/q}t.$$

By applying the q-Hölder inequality, we have

$$|R_n(f;q)| \leqslant \left[\int_0^b \left|\left(D^{m+1}_{1/q}f\right)(q^m t)\right|^{p_1} d_{1/q}t\right]^{\frac{1}{p_1}} \left[\int_0^b |K(x,t)|^{p_2}\, d_{1/q}t\right]^{\frac{1}{p_2}}$$

for all $1 \leqslant p_1, p_2 \leqslant \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Since the second integral in the above equation is independent of f, by choosing coefficients and nodes appropriately we can minimize the remainder.

(i) For $p_1 = \infty$ and $p_2 = 1$,

$$|R_n(f;q)| \leqslant ||D^{m+1}_{1/q}f||_\infty \int_0^b |K(x,t)|d_{1/q}t$$

(ii) For $p_1 = p_2 = 2$,

$$|R_n(f;q)| \leqslant ||D^{m+1}_{1/q}f||_2 \left[\int_0^b |K(x,t)|^2 d_{1/q}t\right]^{\frac{1}{2}}.$$

The Peano kernel $K(x,t)$ can be written as

$$K(x,t) = q^{m(m+3)/2}\frac{(b-\frac{t}{q})^{m+1,q}}{[m+1]_q!} - s(t;q),$$

where $s(t;q) = \dfrac{q^{m(m+1)/2}}{[m]_q!}\sum_{k=0}^{n}\gamma_{nk}(t_{nk}-t)_+^{m,q}$ is a quantum spline with the knot sequence $\{t_{nk}\}_{k=0,\ldots,n}$. Eventually, the problem of minimizing the q-integral

$$\left[\int_0^b |K(x,t)|^{p_1} d_{1/q}t\right]^{\frac{1}{p_1}}$$

is equivalent to finding the best approximation of the polynomial $q^{m(m+3)/2}\dfrac{(b-\frac{t}{q})^{m+1,q}}{[m+1]_q!}$ in t by a quantum spline with respect to the norm $||.||_{p_1}$.

5 Application to divided differences

For about a half century, B-splines have played a central role in approximation theory, geometric modeling and wavelets. Recently their q-analogues or quantum B-splines has been introduced and studied in [12], [3].

In this section we establish certain relations between q-B-splines and q-Peano kernels. When $q=1$, Theorem 5.1 reduces to its classical counterpart which can be found in [10].

The work [3] finds that q-B-splines of degree n are essentially divided differences of q-truncated power functions. That is, the q-B-splines are given by

$$N_{k,n}(t;q) = (t_{k+n+1}-t_k)[t_k,\dots,t_{k+n+1}](x-t)_+^{n,q}.$$

Now recall the fact that a divided difference $f[t_0,t_1,\dots,t_{n+1}]$ can be represented as symmetric sum of $f(t_j)$, see [10],

$$f[t_0,t_1,\dots,t_{n+1}] = \sum_{i=0}^{n+1} f(t_i)/\prod_{\substack{j=0\\j\neq i}}^{n+1}(t_i-t_j). \tag{3}$$

Hence we can readily derive

$$N_{k,n}(t;q) = (t_{k+n+1}-t_k)\sum_{i=k}^{k+n+1}(t_i-t)_+^{n,q}/\prod_{\substack{j=k\\j\neq i}}^{k+n+1}\frac{1}{(t_i-t_j)}$$

The following theorem is also derived in [3] by a different method.

Theorem 5.1.

$$f[t_0,t_1,\dots,t_{n+1}] = \frac{q^{n(n+1)/2}}{[n]_q!}\int_a^b \frac{N_{0,n}(t;q)}{t_{n+1}-t_0}\left(D_{1/q}^{n+1}f\right)(q^n t)d_{1/q}t.$$

Proof. We first set L as

$$\begin{aligned} f[t_0,t_1,\dots,t_{n+1}] &= \sum_{i=0}^{n+1} f(t_i)/\prod_{\substack{j=0\\j\neq i}}^{n+1}(t_i-t_j) \\ &= L(f). \end{aligned}$$

We see that, for any fixed and distinct points $\{t_i : i=0,1,\dots,n+1\}$, L is a bounded linear operator. From the q-Peano Kernel Theorem 3.2, we have

$$L(f) = \int_a^b K(x,t)(D_{1/q}^{n+1}f)(q^n t)d_{1/q}t,$$

where

$$\begin{aligned} K(x,t) &= \frac{q^{n(n+1)/2}}{[n]_q!}L\left((x-t)_+^{n,q}\right) \\ &= \frac{q^{n(n+1)/2}}{[n]_q!}\sum_{i=0}^{n+1}(t_i-t)_+^{n,q}/\prod_{\substack{j=0\\j\neq i}}^{n+1}(t_i-t_j). \end{aligned}$$

Thus

$$K(x,t) = \frac{q^{n(n+1)/2}}{[n]_q!}\frac{N_{0,n}(t;q)}{t_{n+1}-t_0}.$$

Combining the last equation with (3) we derive

$$f[t_0,t_1,\dots,t_{n+1}] = \frac{q^{n(n+1)/2}}{[n]_q!}\int_a^b \frac{N_{0,n}(t;q)}{t_{n+1}-t_0}\left(D_{1/q}^{n+1}f\right)(q^n t)d_{1/q}t.$$

□

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